Teaching Algebra Conceptually: Student Achievement

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This paper reports findings from the second year of a two-year study designed to develop approaches to teaching algebra in years 9 and 10. The aim of the research was to explore and develop teaching approaches to assist students to acquire a conceptual understanding of algebra, and to document the impact of these approaches on student outcomes. Previous work (Linsell, 2009) has detailed the strategies that students use to solve equations but there has been no research to date on teaching approaches that make use of these findings. The two main components of this study were a qualitative description of teaching approaches based on videos, teachers' diaries and meeting notes, and a quantitative analysis of student outcomes based on results from a previously developed assessment tool (Linsell, Savell, Johnston, Bell, McAuslan, & Bell, 2006). Approaches to teaching were not uniform between teachers, but all made extensive use of the diagnostic assessment information and recognised that algebraic thinking pervades the entire mathematics curriculum. Assessment of the students at the end of either one or two year's participation in the study showed significant improvements in algebraic strategies and knowledge compared to baseline, though there was little improvement during the second year. Furthermore, the measures of student outcomes displayed significantly higher values for students in this study compared to year 9 and year 10 students in Secondary Numeracy Project (SNP) schools, which are fairly representative of the population.

A large-scale study by Warren (2003) found that the majority of students in Queensland leave primary school with a limited awareness of mathematical structure and of arithmetic operations as general processes. Bednarz, Kieran, and Lee (1996) have shown that algebra in schools is often reduced to rules for transforming and solving equations, and Brekke (2001) has shown that these rules are often meaningless for students. In New Zealand at the secondary school level, the emphasis within the Number and Algebra strand shifts from number towards algebra. The development of algebraic thinking at years 9 and 10 is therefore of great importance. The Best Evidence Synthesis (BES) project (Anthony & Walshaw, 2007) identified outcomes-linked evidence about what works in the secondary school context as a gap in our knowledge.

This project is an extension of a Teaching and Learning Research Initiative (TLRI) study carried out in 2006, and work undertaken in 2007 – 2010, funded by the New Zealand Ministry of Education. It makes use of the diagnostic assessment tool developed in the previous TLRI study (Linsell, et al., 2006) and results about student achievement from the Ministry of Education Secondary Numeracy Project (SNP) study (Linsell, 2010). Previous work detailed the strategies that students use to solve equations and the prerequisite knowledge required for implementing these strategies. However there has been no research to date on teaching approaches that make use of these recent findings other than preliminary

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results from the first year of this study (Linsell, Tozer, Anakin, Cox, Jones, McAuslan, Smith, & Turner, 2011).

The current study is based on a conception of algebraic thinking as awareness of mathematical structure, rather than a collection of rules and procedures to be learnt. Mason, Stephens, and Watson (2009, p10) define mathematical structure to be "the identification of general properties which are instantiated in particular situations as relationships between elements or subsets of elements of a set." With this definition in mind, our study then takes a similar approach to the New Zealand Numeracy Development Projects (NDPs) by separating the algebraic strategies that students use from the prerequisite knowledge required, and by designing learning experiences that address the identified needs of the students. The results from this project help to address the gaps in knowledge about successful approaches to teaching algebra in secondary schools.

Methodology

The research was carried out over two years and the approach used employed both qualitative and quantitative methods. For impact on student outcomes, pre/post data were used as well as comparisons with a representative sample of the population, the year 9 and 10 students from Linsell's (2010) SNP study. For teaching approaches, case studies were employed. The research questions addressed were:

- How can algebra diagnostic assessment information be used for designing differentiated learning at years 9 and 10?
- What is the impact of the teaching approaches on the most sophisticated strategies students are able to use to solve equations, find relationships between variables, and express generality?
- What is the impact of the teaching approaches on students' understanding of inverse operations, arithmetic structure, equivalence, acceptance of lack of closure and knowledge of algebraic notation and conventions?

Subjects

The study took place in one secondary school and one year 7 to 13 college, both in New Zealand, with five teachers participating. During 2011 a total of 185 students took part in the study. However, this paper presents results only from 107 year 9 students, and 46 year 10 students who also participated as year 9 students in 2010.

Assessment Tools

An extended version of the diagnostic assessment tools developed by Linsell, et al (2006) was used to document the algebraic thinking of the students. A full description of the assessments and scoring rubrics for students' strategies for solving equations, understanding of algebraic notation and convention, inverse operations, arithmetic structure, equivalence, and acceptance of lack of closure is given in Linsell (2010). During 2011 additional questions were added to investigate students' strategies for finding relationships between variables and expressing generality. These additional aspects of students' algebraic thinking, which are consistent with Trigueros and Ursini's (2001) model of three uses of variables, were included because the 2010 tool was considered not to encompass the full range of algebraic thinking.

Teaching Approaches

Much of the project was spent designing, trialling, and developing teaching approaches that were specific to the learning needs of the students. As well as considering the most sophisticated strategies that students were able to use, teaching addressed the prerequisite knowledge required by the students. In order to develop teaching approaches Borko's (2004) model of professional development was employed, which involved teachers capturing their lessons on video and sharing selected excerpts with the others in the project. There were regular meetings of the group, in which the experiences of the teachers were used to refine the teaching approaches for the other teachers to try.

Data Analysis of Student Achievement

The measures of student achievement investigated were strategies for solving equations, finding relationships between variables, expressing generality, and knowledge of algebraic notation and conventions, inverse operations, arithmetic structure, equivalence, and acceptance of lack of closure. For each of these measures a score was generated for each student at baseline and end of year. No assumptions were made about normality of data and differences between the baseline and end of year data were tested using the Wilcoxon signed ranks test. Differences between the end of year data and control groups of students at the same year levels in the SNP study were tested using the Mann-Whitney U-test.

Results and Discussion

Teaching Approaches

The teaching approaches developed in the study were not uniform, but were responsive to the particular needs of each class of students and also reflected the teaching styles and beliefs of the teachers involved. Differentiated learning was approached in quite different ways in the two schools. One school used mixed ability classes and created individualised learning for students through use of laptops. The other school streamed classes according to ability and the learning experiences for each class were designed to address the level at which students were working. There was, however, a great deal in common and a consensus was achieved on effective ways to help students learn algebra. Effective teaching approaches involved: establishing what each student understood through assessment, showing students how algebra is everywhere in our lives, promoting algebraic thinking with rich and meaningful contexts, demonstrating the benefits of rigour and correct use of vocabulary, and building a 'toolbox' of knowledge and skills that students could use appropriately and efficiently.

Assessment. The use of diagnostic assessment was considered to be essential so that decisions could be made about next learning steps, especially as students came from a number of contributing schools and therefore had widely differing prior experiences. The purpose was to document the most sophisticated strategies that a student could use and what prerequisite knowledge they had, not to generate an overall score. There was considerable discussion about the arithmetic skills and knowledge that many high school teachers might assume that their students would have learnt at primary school. Consistent with the findings of Warren (2003), the diagnostic assessment revealed that many students did not have a good understanding of arithmetic structure, inverse operations or equivalence. Students' knowledge of algebraic notation and conventions and acceptance of lack of closure were also documented, as well as their strategies for solving equations, expressing generality and finding relationships between variables. Teachers' approaches with their classes and interventions with individual students throughout the project were guided by the detailed knowledge they had acquired.

Algebra everywhere. A major plank of the approaches taken was to reject the common practice of teaching algebra as isolated units of work once a term or, even worse, once a year. Instead the teachers integrated the teaching of algebraic thinking throughout their programmes. For example, when teaching equivalent fractions such as $\frac{3}{4} = \frac{9}{12}$ and $\frac{55}{25} = \frac{11}{5}$ by multiplying or dividing numerators and denominators by the same factor, this was generalised to $\frac{ac}{ab} = \frac{c}{b}$. The generalisation and the use of algebraic notation to describe the generalisation, was made explicit to the students. A great deal of emphasis was placed on the patterns found when calculating with rational numbers and with rational expressions. Furthermore, the algebraic skills taught were used throughout the year in a variety of contexts and reinforced through regular maintenance.

Rigour and vocabulary. Observation of students' work and discussions with them indicated that very informal written working and a lack of correct mathematical vocabulary were impediments to good mathematical practice and engagement in mathematical discourse. The teachers were particularly careful with correct use of mathematical vocabulary throughout the year. Any terms that were unfamiliar or ambiguous to the students were defined and written into the students' notebooks. Careful board work that modelled correct setting out was used and similar standards were insisted on from the students. Conventions of notation were not assumed but were explicitly taught. Similarly, mathematical identities and laws were made explicit and expressed as generalisations using algebraic notation. The schools ensured that there was consistency from teacher to teacher and from one year level to the next.

The toolbox. When algebraic skills were identified, they were presented to the students as tools to put in their toolboxes. This metaphor was used to promote acceptance and understanding of the skills. These skills included, but were not limited to substitution, manipulating terms, expanding brackets, factorising, and strategies for solving equations. Number skills related to indices, integers, order of operations, basic facts, squares, cubes, and highest common factors were also placed in the students' toolboxes. When solving problems in any context, students were encouraged to select and use tools purposefully. This approach avoided skills being taught in isolation, as students appreciated that these were tools that they would use frequently. Thus the skills became more readily transferrable.

Context. It was regarded as essential that students should perceive algebra as being meaningful and a wide variety of contexts were used to ensure this. Furthermore, teachers interpreted and used the idea of context in a number of different ways. When solving problems in areas such as rational number, students were encouraged to use modelling approaches and set up equations to solve. Science teachers were consulted to ensure that approaches to solving equations and using notation were consistent between mathematics and science lessons. Also, some of the teachers chose to develop thematic units aimed at developing algebraic concepts in holistic ways.

Student Achievement

Strategies for solving equations. Previous work with SNP students (Linsell, 2010) has shown that many students are restricted to very unsophisticated strategies, such as guess and check, and may not even be able to use inverse operations for simple equations of the form

$ax = b$. Furthermore, very few students at the end of years 9 and 10 are able to solve equations formally by carrying out transformations (i.e., by doing the same thing to both sides). Baseline measurements in this project were consistent with these previous findings. Summative assessment at the end of the year, as shown in Figure 1, showed highly significant improvement from baseline in year 9 students' use of strategies for solving equations ($p<0.001$, Wilcoxon signed ranks test). Furthermore, the year 9 students were significantly more sophisticated in their use of strategies than their counterparts in the SNP study ($p<0.001$, Mann-Whitney U-test).

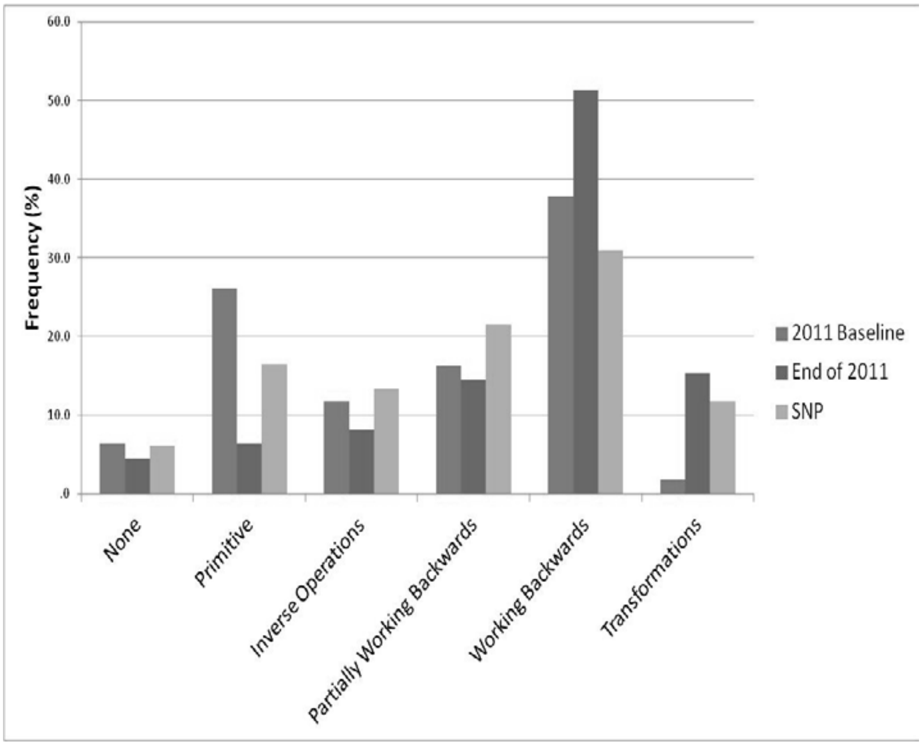

Figure 1. Year 9 students' most sophisticated strategy for solving equations.

Similarly, the year 10 students were significantly more sophisticated in their use of strategies than their counterparts in the SNP study ($p<0.001$, Mann-Whitney U-test). The major difference between the SNP students and these was that after two years in this project very few year 10 students were using unsophisticated strategies and most were solving equations of forms $ax + b = c$, $\frac{x-a}{b} = c$, etc., by working backwards or by using transformations. However, while 60% of them were able to use the strategy of working backwards, only 26% of them were able to use the most sophisticated strategy of carrying out transformations by doing the same thing to both sides, even at the end of two years' participation. For these year 10 students, a rather interesting picture emerged. Even though there was a highly significant improvement in their use of strategies for solving equations during their first year in the project ($p<0.001$, Wilcoxon signed ranks test), there was no significant improvement during the second year ($p=0.24$, Wilcoxon signed ranks test.

Algebraic knowledge. At each assessment point there were four questions in each area that were considered to be prerequisite algebraic knowledge. Each student therefore had a score between zero and four on arithmetic structure, inverse operations, algebraic notation and convention, acceptance of lack of closure, and understanding of equivalence.

For the year 9 students all of the measures showed significant improvements from baseline to end of year assessment ($p<0.001$, Wilcoxon signed ranks test), except for knowledge of inverse operations ($p=0.83$). These year 9 students' scores on four of the measures of algebraic knowledge at end of year were significantly higher than their counterparts in the SNP study ($p<0.01$, Mann-Whitney U-test), again with knowledge of inverse operations showing no significant difference.

For the year 10 students only knowledge of notation and convention ($p<0.01$, Wilcoxon signed ranks test) and understanding of equivalence ($p<0.05$) showed significant improvements from start of year to end of year assessment, and knowledge of inverse operations actually showed a decrease ($p=0.001$). However, all of the measures were still significantly higher than in their counterparts in the SNP study ($p<0.001$ for four measures and $p<0.05$ for knowledge of inverse operations, Mann-Whitney U-test). As with the development of strategies for solving equations, most of the improvement in algebraic knowledge for the year 10 students had occurred in their first year in the project.

The apparent lack of improvement over the year in knowledge of inverse operations for the year 9 students, and decreased score for year 10 students is difficult to account for. The questions used in the pre and post tests and in the SNP test appear to be quite equivalent. Furthermore the teachers were very aware of the gaps in their students' knowledge and actively addressed them in their teaching.

Strategies for finding relationships between variables. It is well known that students find it difficult to determine relationships of the form $y = ax + b$ when presented with tables of values (Hoyles & Küchemann, 2001; Trigueros & Ursini, 2001). We therefore included easier questions addressing additive and directly proportional relationships, and also questions that allowed students to find and make use of relationships even if they were not able to express them algebraically. Factor analysis of data from the whole 2011 cohort revealed four factors, which we described as *additive strategies, multiplicative strategies, use of algebraic conventions,* and *formal use of linear relationships*. Nine questions correlated strongly ($r>0.65$) with one of these factors on both the baseline and summative tests, and were used to indicate to indicate the strength of thinking on these factors. Questions correlating strongly with the additive strategies factor could be solved by purely arithmetic additive strategies. Questions correlating strongly with the multiplicative strategies factor could be solved by purely arithmetic multiplicative strategies. Questions correlating strongly with the use of algebraic conventions factor required students to determine relationships of the form $y = x + a$ or $y = ax$ from a table of values. Questions correlating strongly with the factor formal use of linear relationships required students to determine relationships of the form $y = ax + b$ from a table of values, or to use the relationship even if they were not able to express it formally.

For the year 9 students the scores for additive strategies were very high at baseline and showed no significant change at end of year assessment. There were, however, significant improvements in scores for use of multiplicative strategies ($p<0.05$), use of algebraic conventions ($p<0.001$), and formal use of linear relationships ($p<0.001$, Wilcoxon signed ranks test). However, at the end of the year and in spite of the improvement in formal use of linear relationships, only 29% of year 9 students were able to determine a relationship of the form $y = ax + b$ when presented with a table of values.

For the year 10 students there was no significant improvement during the year on any of the scores other than use of algebraic conventions ($p<0.01$, Wilcoxon signed ranks test).

Strategies for expressing generality. It has been known for a long time that students find it difficult to generate algebraic expressions for structured situations and operate on them (Lee, 1987). Most work with high school students has focussed on situations that give rise to linear or quadratic expressions (see, for example, Mason, Graham, & Johnston-

Wilder, 2005), so we decided to also include easier situations that could be described by expressions such as $x + a$ or ax.

At baseline 58% of year 9 students were unable to generate even simple expressions such as $x + a$ or ax. By the end of the year there were significant increases in their ability to generate additive ($p<0.001$), multiplicative ($p<0.001$), and linear expressions ($p<0.001$, Wilcoxon signed ranks test). However, there was no significant change in their ability to operate on generalised numbers to generate non-linear expressions, with only 13% at end of year 9 able to do so.

For the year 10 students there was no significant change in their ability to generate additive, multiplicative or non-linear algebraic expressions, but a significant decrease in their ability to generate linear expressions ($p<0.001$, Wilcoxon signed ranks test).

Conclusions

This study has demonstrated that achievement in simple algebra can be enhanced by taking a structural perspective that provides high quality diagnostic assessment information to teachers and that integrates algebra into the curriculum, rather than treating it as a separate topic. Through the use of a diagnostic tool which reveals students' algebraic knowledge and strategies, teachers were able to gain significant insights into their students' algebraic world. This information was empowering for the teachers. They placed importance on the diagnostic assessment information to inform them of their students' strengths and weaknesses and in addressing these, they encouraged their students to 'see algebra everywhere' and provided them with relevant and meaningful contexts for exploration.

Many students had poor understandings of arithmetic structure, inverse operations, and equivalence. The teachers in this study addressed these gaps in knowledge to assist the students to use the more sophisticated strategies for solving equations, finding relationships between variables, and expressing generality.

For the year 9 students, the teaching approaches used in this project appear to have improved their prerequisite knowledge and enabled them to progress from being restricted to the least sophisticated strategies. However, even when students have a firm foundation in arithmetic, the most sophisticated algebraic strategies are clearly very difficult to master. Relatively few year 9 students were able to use the most sophisticated strategies of using transformations for solving equations, formal linear relationships for finding relationships between variables, or operating on generalised numbers to generate non-linear expressions. These strategies have in common the requirement for students to operate on the unknowns, generalised numbers or variables.

There was very little progress for the year 10 students during the second year of their participation in the project, in spite of them having made good progress during year 9. The apparent ceiling reached by many students is of great concern.

The teaching approaches used do not appear to have facilitated transition across the 'cognitive gap' between arithmetic and algebra identified by Herscovics and Linchevski (1994). They described this gap as being characterised by "the student's inability to operate with or on the unknown" (p75). It can be argued that the less sophisticated strategies being mastered by the students in this study are essentially arithmetic. Teachers should not underestimate the conceptual leap required for operating on generalised numbers, unknowns and variables. Further research is required on teaching approaches that may facilitate students' transitions to the more sophisticated strategies.

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